Abstract
To transport something in an optimal manner - this is a problem people faced for hundreds and thousands of years. But just recently - in the 1940s - Kantorovich tried to find a solution to this problem by modeling it from a mathematical point of view. He formulated ‘Kantorovich’s optimal transportation problem’. As a linear minimization problem with convex constraints, it admits a dual formulation which is celebrated as the Kantorovich duality. However, he only focused on the two marginal setting. The generalization of this famous duality to multi-marginal optimal transportation creates far more ways to apply this topic in economics.

Theory of optimal transportation
It is a common occurrence that children have the great idea to build their sandcastles in the grass instead of the sandbox. Then their parents face the task of restoring the initial situation in the most efficient manner possible. This problem is the perfect approach to introduce the theory of optimal transportation. Both the pile and the hole in the sandbox have the same volume which can be normalized to 1. Hence, they can be modeled by probability measures \( \mu \) and \( \nu \) respectively.

Historically Monge presented the initial approach. He captured the given problem in the following formulation:

Monge’s optimal transportation problem
Minimize
\[
I(T) = \int c(x, T(x))d\mu(x)
\]
over the set of all measurable \( T \) such that \( T \# \mu = \nu \).

Kantorovich built upon Monge’s results, however he changed one vital aspect. He left himself a loophole enabling him to split mass by using probability measures on the product space \( X \times Y \) as transference plans. All of the measures on the product space are summarized in the set \( \mathcal{P}(X \times Y) \). As it is necessary that \( \mu \) is transported to \( \nu \), only those transference plans are admissible, which are an element of

\[
\Pi(\mu, \nu) = \{ \pi \in \mathcal{P}(X \times Y) | \pi(A \times Y) = \mu(A), \pi(X \times B) = \nu(B) \}
\]
for all measurable \( A \subseteq X \) and \( B \subseteq Y \).

This definition at hand one can move onto considering:

Kantorovich’s optimal transportation problem
Minimize
\[
I(\pi) = \int_{X \times Y} c(x, y)d\pi(x, y)
\]
for \( \pi \in \Pi(\mu, \nu) \).

Kantorovich’s optimal transportation problem is a linear minimization problem with convex constraints. Therefore it admits a dual formulation. In the considered case, this formulation is the celebrated Kantorovich duality. As the optimal transportation problem itself can be transferred in the multi-marginal setting in a natural manner, one is tempted to translate the dual problem as well. Then the multi-marginal duality provides an important alternative formulation to the transportation problem. Instead of minimizing the functional \( I \) over all probability measures with fixed marginals, one can maximize a certain dual problem over \( n \) potentials, each of which is weighted with one of the marginals.

Multi-marginal optimal transportation
Theorem: Generalized Kantorovich Duality
Let \( X_1, \ldots, X_n \) be Polish spaces and \( \mu_1, \ldots, \mu_n \) be Borel-probability measures on \( X_1, \ldots, X_n \) respectively. Moreover, assume that \( c \colon X_1 \times \ldots \times X_n \to \mathbb{R} \cup \{+\infty\} \) is a lower semi-continuous cost function.

Now we define
\[
\forall x \in P(X_1 \times \ldots \times X_n) : f(\pi) = \int_{X_1 \times \ldots \times X_n} c(x_1, \ldots, x_n)d\pi(x_1, \ldots, x_n).
\]

Multi-marginal optimal transportation and applications in economics
The generalized Kantorovich duality is a great tool to tackle economic problems. In the following the market for houses is captured within the mathematical objects of the Kantorovich duality.

Economic applications
The generalized Kantorovich duality is a great tool to tackle economic problems. In the following the market for houses is captured within the mathematical objects of the Kantorovich duality.

<table>
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<th>mathematical formulation</th>
<th>economic interpretation</th>
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<tr>
<td>( f(\pi) = \inf_{\pi \in \Pi(\mu_1, \ldots, \mu_n)} I(\pi) )</td>
<td>These are the costs the HBC has to pay for ( x \times X_1 )</td>
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