Let $k$ be an algebraically closed field of characteristic $p \geq 0$. Both canonical surface singularities and certain simple algebraic groups are classified by simply laced Dynkin diagrams. Quite surprisingly, under some restrictions on $p$, it turns out that there is a more fundamental relation between these two kinds of objects, which was discovered by Brieskorn [2] ($p = 0$), Slodowy [5] ($p \gg 0$ very good) and Shepherd-Barron [3] ($p$ very good). More precisely, the nilpotent variety of the Lie algebra of a simple algebraic group restricted to a transverse slice to the orbit of a subregular nilpotent element under the adjoint action is a rational double point of the corresponding Dynkin type. In turn, this relation can be used to study deformations of rational double points and their simultaneous resolutions using Grothendieck’s resolution of the adjoint quotient.

### Rational double points

**Definition/Theorem.** A normal surface singularity $(X, x)$ is called a rational double point if it satisfies one of the following equivalent conditions.

(i) $(X, x)$ is a canonical surface singularity.

(ii) $(X, x)$ is rational of multiplicity $\mu_k(X) = 2$.

(iii) $(X, x)$ is rational of embedding dimension $\dim_e(X) = 3$.

(iv) $(X, x)$ is rational and Gorenstein.

(v) $(X, x)$ is of multiplicity $\mu_k(X) = 2$, of embedding dimension $\dim_e(X) = 3$ and it can be resolved by successive blow-ups of points.

(vi) The completion $\hat{k}(x, y, z)_x$ is isomorphic to $k[x, y, z]/f_p$, where $f_p$ is one of the equations in the table below if $p \not\equiv 2, 3, 5 \mod 6$ (see [1] for the equations if $p \in \{2, 3, 5\}$).

(vii) The dual graph of the minimal resolution of $(X, x)$ is a Dynkin diagram of type $A_n, D_n, E_6, E_7, E_8$ or $E_{6,7,8}$. Moreover, all irreducible components of the exceptional locus are isomorphic to $\mathbb{P}^1$ and have self-intersection $-2$.

### Simple algebraic groups

**Definition.** A linear algebraic group $G$ is called simple if $G$ has no closed connected normal subgroups other than $G$ itself and the trivial subgroup.

**Notation.** In the following, we denote by $G$ a simple algebraic group, by $B$ a Borel subgroup of $G$, and by $T \subseteq B$ a maximal torus of $G$ with Weyl group $W$. Moreover, let $g, b$ and $t$ be the corresponding Lie algebras.

Using the action of $W$ on the characters of $T$, one obtains an indecomposable root system $\Gamma$ and we call $\Gamma$ simply laced if all its roots have the same length. Conversely, any indecomposable root system arises in this way (Chevalley). Simple algebraic groups giving rise to the same root system $\Gamma$ are related by isogenies and there are unique "biggest" and "smallest" such groups, the simply connected resp. adjoint simple group of type $\Gamma$. Thus, up to isogeny, simple algebraic groups are classified by indecomposable root systems.

**Theorem.** Any simply laced and indecomposable root system is of one of the following types: $A_n, D_n, E_6, E_7, E_8$ with corresponding Dynkin diagrams.

### Classification

<table>
<thead>
<tr>
<th>Equation $f_p$ for $p \not\equiv 2, 3, 5$</th>
<th>Resolution graph</th>
<th>Type $\Gamma$</th>
<th>Very good primes for $\Gamma$</th>
<th>Simply connected and adjoint simple algebraic group of type $\Gamma$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$z^{p+1} + xy$</td>
<td>$2 \times 2$</td>
<td>$A_n$</td>
<td>$p \equiv 2, 3, 5 \mod 6$</td>
<td>$\text{Sp}(2n,\mathbb{C})$</td>
</tr>
<tr>
<td>$z^2 + x^2 y + y^3 - 2$</td>
<td>$2 \times 1$</td>
<td>$D_n$</td>
<td>$p \equiv 2, 3, 5 \mod 6$</td>
<td>$\text{Spin}(2n)$</td>
</tr>
<tr>
<td>$z^2 + x^3 + y^2$</td>
<td></td>
<td>$E_6$</td>
<td>$p \equiv 2, 3, 5 \mod 6$</td>
<td>$\text{PSO}(2n)$</td>
</tr>
<tr>
<td>$z^2 + x^3 + xy$</td>
<td></td>
<td>$E_7$</td>
<td>$p \equiv 2, 3, 5 \mod 6$</td>
<td>$\text{PSO}(2n)$</td>
</tr>
<tr>
<td>$z^2 + x^3 + y^5$</td>
<td></td>
<td>$E_8$</td>
<td>$p \equiv 2, 3, 5 \mod 6$</td>
<td>$\text{PSO}(2n)$</td>
</tr>
</tbody>
</table>

### Deformations of rational double points in nilpotent varieties of simple Lie algebras

**Definition.** The adjoint representation of $G$ is $\text{Ad}: G \to \text{GL}(\mathfrak{g})$, $x \mapsto d([\text{Int}_x])$, where $d([\text{Int}_x])$ is the differential of conjugation with $x$. The quotient $\varphi: g \to g/\mathfrak{g}$ by this action is called adjoint quotient and $\mathcal{N}(g) := \varphi^{-1}(0)$ is the nilpotent variety.

Let $x \in g$ and $Z_G(x) := \{x \in G|\text{Ad}(g)(x) = x\}$. We call $x$

- *regular* if $\dim(Z_G(x)) = \text{rank}(G)$, and
- *subregular* if $\dim(Z_G(x)) = \text{rank}(G) + 2$.

A transverse slice in $g$ to the orbit of $x \in g$ is a locally closed subvariety $S \subseteq g$ such that $x \in S$, $G \times S \to g$ is smooth and $\dim S$ is minimal.

**Theorem 1** ([2], [5], [3]). Let $G$ be of type $\Gamma$, char($k$) very good for $\Gamma$ and $S$ a transverse slice to the orbit of a subregular, nilpotent $x \in g$ such that $\varphi^{-1}(0) = \{x\}$. Then,

- $\mathcal{N}(g) \cap S$ is a rational double point of type $\Gamma$, and
- $\gamma$ is the minimal deformation of $(\mathcal{N}(g) \cap S, x)$.

### Simultaneous resolutions

**Theorem 2** (Grothendieck’s simultaneous resolution of the adjoint quotient, [4]). Under the assumptions of Theorem 1, there is a commutative diagram with maps

$$
\begin{array}{ccc}
\psi_J : \langle g, b \rangle & \to & \text{Ad}(g)(b) \\
\theta_J : \langle g, b \rangle & \to & B_{\text{st}}
\end{array}
$$

such that it is a simultaneous resolution of $\varphi_J$.

### Questions

- How can we generalize the above correspondence to all characteristics?
- Can we compute invariants of rational double points and their deformations using the connection with algebraic groups?

### References


