Lévy-driven tempo-spatial Ornstein-Uhlenbeck processes
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Abstract
We extend the Lévy-driven Ornstein-Uhlenbeck process as a time-wise process to time and space. This is achieved by employing stochastic Volterra integral equations in time and space, which comprise a stochastic integral with respect to a Lévy basis. We formulate conditions for the existence and uniqueness of the solution and derive an explicit solution formula. After giving criteria for stationarity of these processes, we establish the second order structure in the stationary case by means of this solution formula. The theoretical results are illustrated by concrete examples. For further details we refer to [3].

Motivation
A Lévy-driven Ornstein-Uhlenbeck (OU) process is defined as the unique solution of the stochastic integral equation

\[ X(t) = \int_0^t -\lambda X(s) \, ds + \int_0^t dL(s), \quad t \geq 0, \]

where \( \lambda > 0 \) and \( L \) is a Lévy process, i.e. \( L \) has independent and stationary increments and càdlàg paths (see e.g. [1, Section 4.3]).

Goal: Generalization to a stochastic process in time and space.

Idea: We generalize the defining stochastic integral equation to:

\[ X(t, x) = \int_0^t \int_{\mathbb{R}^d} X(t-s, x-y) \mu(dy, ds) + f(t, x), \]

where \( \mu \) is a measure on \( \mathbb{R}^d \times \mathbb{R}^d \), and \( \Lambda \) is a Lévy basis.

The deterministic convolution Volterra integral equation

\[ X(t, x) = \int_0^t \int_{\mathbb{R}^d} X(t-s, x-y) \mu(dy, ds) + f(t, x). \]  

(2)
The forcing function \( f \) does not depend on \( X \) (see [2, Section 4.1]).

Lemma: Let \( \mu \in M_a(\mathbb{R}^d \times \mathbb{R}^d) \) such that \( \mu(\{0\} \times \mathbb{R}^d) = 0 \). Then there exists a unique measure \( \rho \in M_a(\mathbb{R}^d \times \mathbb{R}^d) \), called the resolvent, such that \( \rho \cdot \mu = \mu \cdot \rho \).

Theorem: Let \( \mu \in M_a(\mathbb{R}^d \times \mathbb{R}^d) \) with \( \mu(\{0\} \times \mathbb{R}^d) = 0 \). Then for every \( f \in L^1_a(\mathbb{R}^d \times \mathbb{R}^d) \) the unique solution in \( L^1_a(\mathbb{R}^d \times \mathbb{R}^d) \) to (2) is

\[ X = f - \gamma \int_0^t X s \, ds \]  

Stochastic integration w.r.t. Lévy bases
Definition: A stochastic process \((\Lambda(B))_{B \in \mathbb{B}}\) is a Lévy basis if
1) for disjoint \( (B_i)_{i \in I} \in \mathbb{B} \) satisfying \( \bigcup_{i \in I} B_i \in \mathbb{B} \), we have \( \Lambda \left( \bigcup_{i \in I} B_i \right) = \sum_{i \in I} \Lambda(B_i) \) a.s.
2) \( \Lambda(B) \) is independent for disjoint \( (B_i)_{i \in I} \in \mathbb{B} \),
3) \( \Lambda(B) \) is a Lévy basis for all \( B \in \mathbb{B} \), such that \( \Lambda(B) = \Lambda(B) \) satisfies the characteristic triplet.

Definition: A measurable function \( h : \mathbb{R}^d \times \mathbb{R}^d \to \mathbb{R} \) is called Lévy integrable if there exists a sequence of simple functions \( h_n \) such that
1) \( h_n \) converges to \( h \) Lebesgue a.e.,
2) \( \int h_n \, d\Lambda \) converges in probability for all \( B \in \mathbb{B} = \{ \mathbb{R}^d \times \mathbb{R}^d \} \).

In this case we define

\[ \int h \, d\Lambda = \lim_{n \to \infty} \int h_n \, d\Lambda. \]

This definition does not specify the class of Lévy integrable functions. However, there are convenient integrability conditions (see [4, Section 2]).

The stochastic convolution Volterra integral equation
Goal: Solve equation (1).
Idea: Imitate the deterministic theory pathwise.

Theorem: Let
1) \( \mu \in M_a(\mathbb{R}^d \times \mathbb{R}^d) \) with \( \mu(\{0\} \times \mathbb{R}^d) = 0 \),
2) \( g \in L^1_a(\mathbb{R}^d \times \mathbb{R}^d) \) be bounded, and
3) \( \Lambda \) be a homogeneous Lévy basis on \( \mathbb{R}^d \times \mathbb{R}^d \) with finite second moments.

Then the unique solution (up to versions) to (1) is given by

\[ X(t, x) = \int_0^t \int_{\mathbb{R}^d} g(s - \delta \rho)(t-s, x-y) \Lambda(dy, ds), \quad (t, x) \in \mathbb{R}^d \times \mathbb{R}^d. \]  

(3)

Example
A Lévy-driven tempo-spatial Ornstein-Uhlenbeck process, We formulate the existence and uniqueness of the solution and derive an explicit solution formula. After giving criteria

\[ X(t, x) = \int_0^t \int_{\mathbb{R}^d} g(s - \delta \rho)(t-s, x-y) \Lambda(dy, ds) \]

(4)

Stationarity
Problem: No stationary solutions so far.
Idea: Modify the stochastic integral equation to:

\[ X(t, x) = \int_0^t \int_{\mathbb{R}^d} X(t-s, x-y) \mu(dy, ds) + \int_0^t g(t-s, x-y) \Lambda(dy, ds) + V(t, x) \]

(5)

Theorem: Under the additional assumption that \( g \cdot (\delta \rho) \in L^1(\mathbb{R}^d \times \mathbb{R}^d) \) is bounded there exists a stochastic process \( X \) on \( \mathbb{R}^d \times \mathbb{R}^d \) such that equation (5) has a unique (up to versions) strictly stationary solution, namely

\[ X(t, x) = \int_0^\infty \left\{ \int g(s - \delta \rho)(t-s, x-y) \Lambda(dy, ds) \right\} \]  

(6)

Second order structure
Theorem: The second order structure of the strictly stationary solution \( X \) is given by

\[ E(X(0, 0)) = \kappa_1 \int \int (g \cdot (\delta \rho))(s, y) \, ds \, dy \]

(7)

\[ \text{cov}(X(0, x), X(0, y)) = \kappa_2 \int \int (g \cdot (\delta \rho))(s, y) \gamma(\delta \rho)(s, x, y, \delta \rho)(s, y, \delta \rho)(s + x - \delta \rho) \, ds \, dy \]

(8)

for all \( t, f, x, z \in \mathbb{R}^d \),

where \( \kappa_1 = b + \int h_{\rho[1,1]} \, d\Lambda \in \mathbb{R}^d \) and \( \kappa_2 = C + C \int \gamma \, d\Lambda \in \mathbb{R}^d \).

Notation
\[ M(\mathbb{R}^d \times \mathbb{R}^d) \]
signed complete Borel measures on \( \mathbb{R}^d \times \mathbb{R}^d \)

References