

## TopMath Essay Topics

*Applicants for summer semester 2022 and winter semester 2022/23 must select one of the following essay topics:*

### 1. FINITE SUBGROUPS OF $SL_2(\mathbb{C})$

The aim of this essay is the classification of finite subgroups of  $SL_2(\mathbb{C})$  up to conjugation. It turns out that there are two series of such groups and three sporadic ones. There exist relationships between this purely group theoretical problem, the five platonic solids, finite and semisimple Lie algebras, Klein's work on invariant theory and quotient singularities, and probably even more subjects, some of which are not yet fully explored. In this essay, you should prove this classification of the finite subgroups of  $SL_2(\mathbb{C})$ , pick some of the relationships sketched above and discuss them.

- K. Lamotke, *Regular Solids and Isolated Singularities*, Vieweg, Braunschweig, Wiesbaden 1986

### 2. EULER'S POLYHEDRON FORMULA

Euler's polyhedron formula relates the number of edges, faces, and vertices of a convex polyhedron. It has several generalizations, for instance to surfaces (involving the genus, the number of holes), to planar graphs, higher dimensional polyhedra, and to more complicated topological spaces (using the Euler characteristic). In your essay, you should give a precise statement of the theorem and explain one of its proofs. Then discuss one or several applications or generalizations.

### 3. JAMES-STEIN ESTIMATION

Let  $X = (X_1, \dots, X_p)$  be a random vector whose coordinates  $X_i$  are independent and normally distributed random variables, with  $X_i \sim N(\theta_i, \sigma^2)$  for  $i = 1, \dots, p$ . Suppose that the variance  $\sigma^2$  is known but that the vector of means  $\theta = (\theta_1, \dots, \theta_p)$  is unknown and to be estimated statistically. Taking  $\hat{\theta}(X) = X$  yields an obvious and unbiased estimator of  $\theta$ . Surprisingly, however, the estimator  $\hat{\theta}(X)$  is not admissible when  $p \geq 3$ . Indeed, Charles Stein and W. James were able to exhibit another estimator  $\hat{\theta}_{JS}(X)$  with strictly smaller mean squared error (MSE), i.e.,  $\mathbb{E}_\theta[\|\hat{\theta}_{JS}(X) - \theta\|^2] < \mathbb{E}_\theta[\|\hat{\theta}(X) - \theta\|^2]$  for all  $\theta \in \mathbb{R}^p$ .

In your essay, present the estimator of James and Stein and give a proof that it indeed improves on  $\hat{\theta}(X)$  in terms of MSE. Discuss whether the James-Stein estimator is admissible and what further improvements may be possible. You may additionally discuss connections to related ideas such as Bayesian estimation or discuss the practical use of the estimators you describe.

## 4. ASYMMETRIC METRIC SPACES

Many theorems in analysis can be phrased in the context of metric spaces. Review results you see as corner stones in this field and investigate which results fail in asymmetric metric spaces (where  $d(x, y) \neq d(y, x)$ ). If possible, give examples of results which fail in asymmetric metric spaces, for example by constructing counterexamples. You can introduce notions of “not quite symmetric but not quite asymmetric either” metric spaces if some results carry over.

*If you have any questions please do not hesitate to contact us via e-mail at [topmath@ma.tum.de](mailto:topmath@ma.tum.de)*