The quantum random energy model (QREM) serves as a simple cornerstone and a testing ground, for a number of fields. It is the simplest of all mean-field spin glass models in which quantum effects due to the presence of a transversal field are studied. Renewed interest in its spectral properties arose recently in connection with quantum annealing algorithms [1,2] and many-body localisation [4,6]. In our paper [5] we prove Goldschmidt’s formula [3] for the QREM’s free energy. In particular, we verify the location of the first order and the freezing transition in the phase diagram. The proof avoids replica methods and is based on a combination of variational methods on the one hand, and percolation bounds on large-deviation configurations in combination with simple spectral bounds on the hypercube’s adjacency matrix on the other hand.

Main Result

Partition function at inverse temperature \( \beta \in [0, \infty) \): \( Z(\beta, \Gamma) = 2^{-N} \text{Tr} e^{-\beta H} \)

**Pressure:** \( p(\beta, \Gamma) = N^{-1} \ln Z(\beta, \Gamma) \)

Thermodynamic limit \( (N \to \infty) \): the pressure of the REM converges almost surely

\[
\lim_{N \to \infty} p_N(\beta, 0) = p_{REM}(\beta) = \begin{cases} \frac{1}{2} \beta^2 & \text{if } \beta \leq \beta_c \\ 0 & \text{if } \beta > \beta_c \\ \end{cases}
\]

Freezing transition at inverse temperature \( \beta = \sqrt{2 \ln 2} \); \( \beta_c \) coincides with specific ground state energy.

Paramagnetic pressure \( (U = 0) \): \( p_{PBM}(\beta) = \ln \cosh(\beta) \)

**Theorem M./W. ’19** For any \( \Gamma, \beta \geq 0 \), we have the almost sure convergence

\[
\lim_{N \to \infty} p_N(\beta, \Gamma) = \max \left\{ p_{REM}(\beta), p_{PBM}(\beta) \right\}
\]

Goldschmidt calculated the limit of the pressure via the (non-rigorous) replica method and static approximation in path-integral representation of \( E[Z(\beta, \Gamma)] \).

Sketch of the Proof

**Basic idea:** prove pair of asymptotically coinciding upper and lower bound for \( p_N(\beta, \Gamma) \).

1. **Lower bound:** Based on Gibbs variational principle

\[
\ln \text{Tr} e^{-\beta H} = \inf_{\rho \in \text{density matrix}} \left\{ \beta \text{Tr}(H \rho) + \text{Tr} \left( \rho \ln \rho \right) \right\}
\]

Pick REM Gibbs state \( \rho = e^{-\beta H}/\text{Tr} e^{-\beta H} \) and paramagnetic Gibbs state \( \rho = e^{-\beta g}/\text{Tr} e^{-\beta g} \):

\[
p(\beta, \Gamma) - p_{REM}(\beta) \geq - \frac{1}{N} \text{Tr} e^{-\beta H} \\
p_N(\beta, \Gamma) - p_{REM}(\beta) \geq - \frac{1}{N} \text{Tr} e^{-\beta g} \\
\Rightarrow \liminf_{N \to \infty} p_N(\beta, \Gamma) \geq \max \left\{ p_{REM}(\beta), p_{PBM}(\beta) \right\}
\]

2. **Upper bound:** Consider for \( \epsilon > 0 \) large deviation set

\( \mathcal{C}_\epsilon \equiv \{ \sigma : \mathbb{Q}_N(U(\sigma)) \leq \epsilon N \} \)

Subset \( \mathcal{C}_\epsilon \subset \mathcal{L} \) called edge-connected pair \( \sigma, \sigma' \in \mathcal{C}_\epsilon \) connected through an edge-path of adjacent edges.

Decompose \( \mathcal{L} = \bigcup \mathcal{C}_\epsilon \) into maximal edge-connected subsets \( \mathcal{C}_\epsilon \).

Decomposition of the Hamiltonian

\( H = U + \Gamma T \)

\( U_L \) and \( H_L \) restrictions of corresponding operators and \( \mathcal{A}_\epsilon \) is remainder term with matrix elements

\[
\langle \sigma | A_\epsilon | \sigma' \rangle = \begin{cases} 1 & \text{if } \sigma \in \mathcal{L}_\epsilon \text{ or } \sigma' \in \mathcal{L}_\epsilon \text{ and } d(\sigma, \sigma') = 1 \\ 0 & \text{else.} \end{cases}
\]

Upper bound for the operator norm \( \| A_\epsilon \| \leq \sqrt{N \max_{\sigma} |\mathcal{C}_\epsilon|} \)

To conclude the upper bound, pick some \( \epsilon > 0 \). The Golden-Thompson inequality yields

\[
Z(\beta, \Gamma) \leq 2^{-N} \text{Tr} e^{-\beta H_L} + \text{Tr} e^{-\beta g_L} \leq e^{\beta \epsilon} \text{Tr} e^{-\beta H_L} + \text{Tr} e^{-\beta g_L}
\]

First term in the bracket: bounded by \( Z(\beta, 0) \)

Second term: all matrix elements of \( -T \) are positive, this leads to the bound

\[
\text{Tr} e^{-\beta g_L} \leq e^{\beta \epsilon} \text{Tr} e^{-\beta g_L}
\]

On \( \mathcal{L}_\epsilon \) we thus get the following bound for all \( N \) large enough,

\[
p_N(\beta, \Gamma) \leq \max \left\{ p_{REM}(\beta), p_{PBM}(\beta) \right\} + 2\beta \epsilon
\]

A Borel-Cantelli argument implies:

\[
\limsup_{N \to \infty} p_N(\beta, \Gamma) \leq \max \left\{ p_{REM}(\beta), p_{PBM}(\beta) \right\}
\]

References