Lecture 5: Exponentially small splitting of separatrices: the singular case

Multiscale Phenomena in Geometry and Dynamics
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The problem

The rapidly perturbed pendulum

The Hamiltonian system we want to understand:

- Hamiltonian

  \[ H(y, x, \frac{t}{\varepsilon}) = \frac{y^2}{2} + (\cos x - 1) + \mu(\cos x - 1) \sin \frac{t}{\varepsilon} \]

- Equations

  \[ \dot{x} = y \]
  \[ \dot{y} = \sin x + \mu \sin x \sin \frac{t}{\varepsilon} \]

- \( \mu \) is a fixed non small constant.

- We will deal with the problem taking \( \varepsilon \) small, but not \( \mu \).

- We will see that the splitting is exponentially small but the Melnikov function is not the main order of the distance.
Plan to find an asymptotic formula for the distance

- Look for suitable parameterizations of the invariant manifolds.

- Extend these parameterizations near the singularities \( \pm i \frac{\pi}{2} \) and find good approximations of the manifolds. This approximations will not be given by the homoclinic orbit anymore.

- Analyze the difference of the parameterizations in these regions.

- Deduce from this analysis the first order of the difference in the reals.
Parameterizations of the invariant manifolds

As before, we look for

\[ y = \partial_x S^{u,s}(x, t/\varepsilon) \]

Parameterization of the pendulum separatrix:

\[ x_0(\nu) = 4 \arctan(e^\nu), \quad y_0(\nu) = \frac{2}{\cosh \nu}. \]

Reparametrize \( x = x_0(\nu), \quad T^{u,s}(\nu, \tau) = S^{u,s}(x_0(\nu), \tau) \)

Parameterizations of the form

\[ x = 4 \arctan(e^\nu), \quad y = \frac{\cosh \nu}{2} \partial_\nu T^{u,s}(\nu, \tau). \]

For the unperturbed pendulum: \( \partial_\nu T_0(\nu) = \frac{4}{\cosh^2 \nu} \).

\[ \varepsilon^{-1} \partial_\tau T + \frac{\cosh^2 \nu}{8} (\partial_\nu T)^2 - \frac{2}{\cosh^2 \nu} - \mu \frac{2}{\cosh^2 \nu} \sin \tau = 0 \]
• Perturbed pendulum: \( T^{u,s} = T_0 + T^{u,s} \).

• \( h = \partial_v T^u \) is solution of:

\[
h = \partial_v G^u \left( \frac{\cosh^2 v}{8} h^2 - \mu \frac{2}{\cosh^2 v} \sin \tau \right)
\]

Complex domains:

• \( \tau \in \mathbb{T}_\sigma = \{ \tau \in \mathbb{C}/(2\pi\mathbb{Z}) : |\Im \tau| \leq \sigma \} \).

• \( v \in D^u \):

Instead of distance \( \varepsilon \) to the singularities we choose a distance \( \rho \varepsilon \), with \( \rho > 1 \). This will help in the fixed point to have a contraction.
The fixed point argument

We can prove the same results without using $\mu$ small. For $v \in D^u$ with $-V_1 \leq \Re v \leq V$, where $V_1 > 0$ is independent of $\varepsilon$, $\mu$:

$$\left| \cosh^3(v) h^u(v, \tau) \right| \leq b_1 |\mu| \varepsilon$$

$$\left| \cosh^3 (h^u(v, \tau) - \partial_v \mathcal{F}^u(0)) \right| \leq b_1 \frac{|\mu|^2 \varepsilon}{\rho}$$

Consequently:

$$\left| h^u(v, \tau) \right| \leq b_1 |\mu| \varepsilon$$

$$\left| h^u(v, \tau) - \partial_v \mathcal{F}^u(0) \right| \leq b_1 \mu^2 \varepsilon \quad \text{if} \quad v \in \Re, \quad v \leq V$$

$$\left| h^u(v, \tau) \right| \leq b_1 \frac{\mu}{\varepsilon^2} \quad \left| h^u(v, \tau) - \partial_v \mathcal{F}^u(0) \right| \leq \frac{b_1 \mu^2}{\rho \varepsilon^2} \quad \text{if} \quad v \in \mathbb{C}, \quad \text{near} \quad \pm \frac{\pi}{2} i$$

Observe that

$$|\partial_v T_0(v)| \leq K \quad \text{if} \quad v \in \Re, \quad v \leq V$$

$$|\partial_v T_0(v)| \leq K \frac{1}{\varepsilon^2} \quad \text{if} \quad v \in \mathbb{C}, \quad \text{near} \quad \pm \frac{\pi}{2} i$$

The perturbed invariant manifolds are THE SAME SIZE than the unperturbed
The problem

Difference between manifolds $\Delta = T^u - T^s$

- Assume that $\Delta \in \text{Ker} L_0$, $L_0 = \varepsilon^{-1} \partial_\tau + \partial_v$. (This is not true and causes the appearance of logarithms in the asymptotics)
- The Melnikov potential $L(v, \tau) = L(0, \tau - \frac{v}{\varepsilon})$ satisfies $L_0 L = 0$
- $L_0(\Delta - L) = 0$.
- for $(v, \tau) \in [-ir, ir] \times \mathbb{T}_\sigma$ with $r = \frac{\pi}{2} - \varepsilon$.

$$|\partial_v \Delta - \partial_v L| \leq \frac{b_1}{\rho} \frac{|\mu|^2}{\varepsilon^2}$$

- We deduce exponentially small bounds for real values of the variables: $| (\partial_v \Delta - \partial_v L)(v, \tau) | \leq \frac{b_1 \mu^2}{\varepsilon^2} e^{-\frac{\pi}{2\varepsilon}}$, for $(v, \tau) \in [-V, V] \times \mathbb{T}_\sigma$
- But remember that $\partial_v L(v, \tau) = 4\pi \frac{\mu}{\varepsilon} e^{-\frac{\pi}{2\varepsilon}}$
- Both terms are of the same size: no asymptotics
The problem

The inner equation

- The unperturbed homoclinic is not a good leading order for the invariant manifolds near the singularities
- To find a better approximation we perform a change of variables: 
  \[ v = i \frac{\pi}{2} + \varepsilon z. \]
- Observe that when \( v - i \frac{\pi}{2} = \mathcal{O}(\varepsilon) \), we know that \( T_0 \) and \( T^u \) are of size \( \frac{1}{\varepsilon} \).
- Therefore we also scale the function \( \phi(z, \tau) = \varepsilon T(i \frac{\pi}{2} + \varepsilon z) \)
- Then \( \partial_z \phi = \varepsilon^2 \partial_v T, \partial_\tau \phi = \varepsilon \partial_\tau T \) and we obtain the equation of \( \phi \):

\[
\frac{\cosh^2(i \frac{\pi}{2} + \varepsilon z)}{8\varepsilon^4}(\partial_z \phi)^2 - \frac{2}{\cosh^2(i \frac{\pi}{2} + \varepsilon z)}(1 + \mu \sin \tau) + \frac{1}{\varepsilon^2} \partial_\tau \phi = 0
\]

- using the asymptotic of the \( \cosh \) and multiplying by \( \varepsilon^2 \) we get:

\[
- \frac{z^2}{8}(1 + \mathcal{O}(\varepsilon z)^2)(\partial_z \phi)^2 + \frac{2}{z^2}(1 + \mathcal{O}(\varepsilon z)^2)(1 + \mu \sin \tau) + \partial_\tau \phi = 0
\]
The inner equation

Putting $\varepsilon = 0$ in the equation one obtains the inner equation:

$$-\frac{z^2}{8} (\partial_z \phi)^2 - \frac{2}{z^2} (1 + \mu \sin \tau) + \partial_\tau \phi = 0$$

As $v = i \frac{\pi}{2} + \varepsilon z$ we have that:

$$\phi^u(z, \tau) = \varepsilon T^u(i \frac{\pi}{2} + \varepsilon z) = \varepsilon T_0(i \frac{\pi}{2} + \varepsilon z) + O(\frac{\mu \varepsilon^2}{\cosh^3(i \frac{\pi}{2} + \varepsilon z)})$$

$$= \varepsilon T_0(i \frac{\pi}{2} + \varepsilon z) + O(\frac{\mu \varepsilon^2}{(\varepsilon z)^3}) = \varepsilon T_0(i \frac{\pi}{2} + \varepsilon z) + O(\frac{\mu}{\varepsilon z^3})$$

$$\partial_v T_0(i \frac{\pi}{2} + \varepsilon z) = \frac{4}{\cosh^2(i \frac{\pi}{2} + \varepsilon z)} = -\frac{4}{(\varepsilon z)^2} (1 + (\varepsilon z)^2)$$

Therefore:

$$\partial_z \phi^u = \varepsilon^2 \partial_v T = -\frac{4}{z^2} + O((\varepsilon z)^2) + O(\frac{\mu}{z^4})$$
The problem

The inner equation

We look for two solutions of the inner equation such that:

\( \phi^{u,s} = \frac{4}{z} + \psi^{u,s} \) and we obtain that

\[
\partial_\tau \psi + \partial_z \psi + \frac{2}{z^2} \mu \sin \tau - \frac{z^2}{8} (\partial_z \psi)^2 = 0
\]

with asymptotic condition (remember \( z = \frac{v - i \frac{\pi}{2}}{\varepsilon} \)):

\[
\psi^u(z, \tau) \to 0, \quad \text{as}, \quad \Im z \leq 0, \quad \Re z \leq 0, \quad |z| \to \infty
\]

and

\[
\psi^s(z, \tau) \to 0, \quad \text{as}, \quad \Im z \leq 0, \quad \Re z \geq 0, \quad |z| \to \infty
\]
The problem

The difference $\Delta \psi = \psi^u - \psi^s$

Next step is to compute the difference $\Delta \psi = \psi^u - \psi^s$. The main point is that $\Delta \psi$ satisfies a linear equation:

$$\partial_\tau \Delta \psi + \partial_z \Delta \psi = \frac{z^2}{8} (\partial_z \psi^u + \partial_z \psi^s) \partial_z \Delta \psi$$

with asymptotic condition:

$$\Delta \psi^u(z, \tau) \to 0, \quad \text{as, } \Im z \leq 0, \quad |z| \to \infty$$

As the term on the right is small, this equation is very close to

$$\partial_\tau \Delta \psi + \partial_z \Delta \psi = 0$$

and the solutions of this equation are $\Delta \psi(z, \tau) = \sum_{k<0} \gamma[k] e^{ik(\tau-z)}$
The problem

The difference $\Delta(v, \tau) = T^u - T^s$

- One can see that:
  \[
  \left| \partial_z \phi^{u,s}(z, \tau) - \partial_z \phi_{0}^{u,s}(z, \tau) \right| \leq K \frac{\varepsilon}{|z|}
  \]
  and therefore:
  - which gives bounds about:
    \[
    \left| \partial_v T^{u,s}(v, \tau) - \frac{1}{\varepsilon^2} \partial_z \phi_{0}^{u,s}(\frac{v - i \frac{\pi}{2}}{\varepsilon}) \right| \leq \frac{K}{|v - i \frac{\pi}{2}|}
    \]
  - Next step is to see that main terms of the difference $\Delta(v, \tau)$ is given by $\Delta \psi$.
  - $\Gamma(v, \tau) = \Delta(v, \tau) - \Delta \psi(\frac{v - i \frac{\pi}{\varepsilon}}{\varepsilon})$ satisfy the linear equation:
    \[
    \frac{1}{\varepsilon} \partial_{\tau} \Gamma - \partial_v \Gamma = 0
    \]
    and we can bound $\partial_v \Gamma$ for $v$ near the singularity
  - This gives exponentially small bounds for $\Gamma(v, \tau)$ for real values of $v$
The problem

The difference $\Delta(v, \tau) = T^u - T^s$

- We obtain an asymptotic formula

$$\partial_v T^u(v, \tau) - \partial_v T^s(v, \tau)$$

$$= \frac{1}{\varepsilon^2} (\partial z \phi_0^u \left( \frac{v - i \pi}{2\varepsilon} \right) - \partial z \phi_0^s \left( \frac{v - i \pi}{2\varepsilon} \right)) (1 + o(1))$$

$$= \gamma[1] \varepsilon^2 e^{i(\tau - z)} + \cdots = e^{-\frac{\pi}{2\varepsilon}} \gamma[1] \varepsilon^2 e^{i(\tau - \frac{v}{\varepsilon})} + \cdots$$

The constant $\gamma[1]$ is called Stokes constant and is different from the constant provided by the Melnikov method.
Conclusions

- In the problem about oscillatory motions the Melnikov function gives the correct asymptotic value of the distance between the invariant manifolds.
- In the Hopf zero singularity the Melnikov function does not give the correct asymptotic value of the distance between the invariant manifolds for generic unfoldings.
- In the perturbed pendulum we have the two different situations depending on the “fake” parameter $\mu$. 