TOPMATH - ESSAY TOPICS

Applicants for summer semester 2021 and winter semester 2021/22 must select one of the following essay topics:

1. Expander graphs

An expander graph is a graph $G = (V, E)$ with high connectivity. For a subset $S \subset V$ of vertices, let $N(S) = \{u \in V \mid \exists s \in S \text{ such that } \{u, s\} \in E\}$ denote the set of neighbors of S. Then G is called a (K, δ)-*vertex-expander* (where $K \in \mathbb{N}$ and $\delta > 0$) if $|N(S)| \geq (1 + \delta) \cdot |S|$ for every subset $S \subset V$ satisfying $|S| \leq K$. We call an infinite family of D-regular graphs a *good* expander if they are expander graphs with $D = O(1)$, $K = \Omega(|V|)$ and $\delta = \Omega(1)$. In your essay, describe why 2-regular graphs are bad expanders, and show (using a probabilistic argument) that good 3-regular vertex expanders exist.

A related concept is that of spectral expansion, which is defined via the spectral gap of the graph. Explain why this notion is relevant for the convergence of a random walk on the graph, and discuss how vertex expansion and spectral expansion can be related. You may additionally explain an application of expander graphs e.g., in computer science.

2. Jensen's inequality and applications

Some mathematicians refer to Jensen's integral inequality as the mother of all estimates, since it is useful in the derivation of so many further inequalities. In your essay, you should review one proof of Jensen's inequality and then discuss the following application. If a function $u = u(t, x)$ satisfies the heat equation $\partial_t u = \partial_{xx} u$ and is such that $u(t, \cdot)$ is integrable with integral zero at each $t \geq 0$, then the map $t \mapsto \int_{\mathbb{R}} \phi(u(t, x))dx$ is monotone for any convex function ϕ with $\phi(0) = 0$. Use the representation of u by means of the heat kernel. Then show that the same statement is in general not true if the heat equation is replaced by its big brother $\partial_t u = -\partial_{xxxx} u$. Discuss - again on the level of solution kernels - why Jensen's inequality does not apply here, and propose some remedy.

3. Classification of Surfaces

The classification of surfaces is a fundamental result in geometric topology. It says that every surface, no matter the shape, is topologically equivalent to some member of a family of surfaces, indexed by the integer numbers.

Give a precise statement of the theorem and provide an outline of one of its proof. In addition, you may discuss further aspects, such as generalizations of the theorem, connected sums of surfaces, or the triangulability of surfaces.

4. A probabilistic lower bound for Ramsey numbers

Ramsey's Theorem, discovered by Frank Plumpton Ramsey in 1930 and published in [2] states that for every natural number k there exists a number R_0 so large that for every $R > R_0$, and for every simple graph of R vertices, there exists either an independent set of size k of a clique (i.e. a subset of vertices s.t. the graph reduced to it is complete) of size k. An important question ever since was to determine (or to estimate) the *Ramsey number* $R(k)$ *of* k which is the smallest number with this property.

One can deduce the upper bound $R(k) \leq 4^k$ directly from Ramsey's original proof. However, for a while, all the known lower bounds were of polynomial growth. Then, in 1947 Erdös proved, and published in [1], that in fact $R(k) > 2^{k/2}$. This is in fact an existence proof - the statement is that there exists a graph of $\lceil 2^{k/2} \rceil$ vertices containing no independent set of size k and no clique of size k. This proof of existence was quite revolutionary in its time in the sense that it contained no construction, but rather a probabilistic calculation showing that a properly chosen random graph of $\lceil 2^{k/2} \rceil$ vertices has positive probability of containing no independent set or a clique of size k. This opened the door to the so called "probabilistic method" in combinatorics, which is one of the most important tools used in modern combinatorics.

In the essay, the student should explain this background and outline Erdös's proof.

References

[1] Erdös, Paul. Some remarks on the theory of graphs. *Bulletin of the American Mathematical society* **53.4** (1947): 292–294.

[2] Ramsey, F. P. On a Problem of Formal Logic. *Proceedings of the London Mathematical Society* **2.1** (1930): 264–286.

If you have any questions please do not hesitate to contact us via e-mail at topmath@ma.tum.de