Abstract
The limiting behaviour of the ground state for certain models of finite rank perturbations of finite dimensional random diagonal potentials is considered. Finite rank perturbation theory is used to reduce the problem to the spectral properties of a meromorphic matrix-valued function of fixed dimension, which is then solvable by regularisation and restriction to appropriate subsets of the probability space. By these means the existence of two regimes for the parameter of disorder is shown. The two regimes are characterised by localisation respectively delocalisation of the ground state. As part of the result precise asymptotics on the ground energy and on a norm-quotient characterising localisation are given. Additional results include a detailed description of the behaviour of the matrix-valued function’s eigenvalues and a resulting statement on the extent of the whole spectrum of the model.

The Model

the model is a sequence of matrices of the following form:

\[ H_N := H_N[\omega] = \sum_{\ell=1}^K \phi_\ell \rho_\ell \left( V(\ell) \right) \left( \phi_\ell \right)^* - T_N \]

where \( \{\left( V(\alpha) \right)_{\alpha \in N(0,1)} \} \) i.i.d. \( N(0,1) \) variables and

\[ T_N = \sum_{j=1}^K a_j \left( \phi_j \right)^* \left( \phi_j \right) \]

cyclic, with \( a_1 \geq a_2 \geq \ldots \geq 0 \), \( \left( \phi_j \right)_{j < N} \) orthonormal and

\[ \kappa_N = \lambda \sqrt{2 \log N} \]

a normalising constant, which guarantees that the spectrum of \( D_N \) typically spans the interval \([-\lambda, \lambda]\). Furthermore a delocalisation assumption for \( T_N \) of the following form is made:

\[ \exists \delta > 0 \text{ s.t. } \forall k \in \mathbb{N} : \|x_k\| \in O(N^{-\delta}) \]

Results

the model exhibits a ground state transition at \( \lambda = 1 \), characterised by a change of the localisation behaviour of the ground state and the asymptotics of its energy.

Theorem 1. Assuming localisation of \( T_N \), the ground energy \( E_N[\omega] \) and corresponding ground state \( \phi_N[\omega] \) satisfy with asymptotically full probability the following asymptotics: if \( \lambda < 1 \):

\[ \begin{align*}
E_N &= -\kappa_N[\omega] - O(\|x_N[\omega]\|) \\
\Omega^2(\lambda^N/4) &\leq \|x_N[\omega]\| \leq O(\lambda^{N/2})
\end{align*} \]

if \( \lambda > 1 \):

\[ \begin{align*}
\|x_N[\omega]\| &\leq 1 + O(N^{-\kappa_N}) \\
E_N &= \min_{x \in \mathbb{N}} \rho_\ell \left( V(\ell) \right) + O(N^{-\delta/4})
\end{align*} \]

where a property \( A(\omega) \) is said to hold with asymptotically full probability (a. f. p.) if

\[ \lim_{N \to \infty} P \left( \text{Pr} \left( \omega \in \Omega | A(\omega) \right) \right) = 1 \]

in that sense the Landau notation is to be understood as:

\[ X_N \in O(\rho(\omega)) \iff \exists C \in \mathbb{R} \text{ mit: } \text{Pr} \left( X_N \geq C \rho(\omega) \right) \xrightarrow{N \to \infty} 0 \]

analogously for \( o(\cdot) \), \( \omega(\cdot) \) etc., i.e. the Landau-Notation holds with a. f. p. with constant \( C \).

Extent of \( \sigma(H_N) \)

by analysing the behaviour of the eigenvalues of \( M_N \) it follows that the spectrum of \( D_N \) and \( H_N \) are intertwined in the sense that the empirical distribution functions of their eigenvalues satisfy \( N(D_N, x) \leq N(H_N, x) \leq N(D_N, x) + K \) this implies that the asymptotic extent of \( \sigma(H_N) \) transfers to \( \sigma(D_N) \) i.e.

\[ \sigma(H_N) \setminus \{E_1, \cdots, E_K\} \xrightarrow{N \to \infty} [-\lambda, \lambda] \text{ in probability} \]

Finite Rank Perturbation Theory

By the means of finite rank perturbation theory the question of the ground state and function can be reduced to the spectral properties of a (random) matrix valued meromorphic function \( M_N : C \to \mathbb{C}^{K \times K} \) with

\[ (M_N)_{ij} := \sum_{\alpha \in \mathbb{N}} \sqrt{\lambda^2 - \alpha^2} \phi_\alpha^i \phi_\alpha^j(x) \]

where the following identity can be shown:

\[ E \in \sigma(H_N) \iff 1 \in \sigma(M_N) \]

The eigenvalues \( \lambda_\alpha(z) \) of \( M_N \) are well behaved as they are meromorphic in \( z \) and (uniformly in the disorder) Lipschitz-continuous away from their singularities.

Probabilistic Considerations

By making multiple assumptions which hold with a. f. p. the uniform (in \( z \) and \( \omega \)) convergence of \( M_N \), a regularised version of \( M_N \), towards \( \rho_\ell \left( V(\ell) \right) \left( \phi_j \right)^* \) on appropriate compact subsets can be shown. Here \( \rho_\ell \) is up to a scaling the Hilbert transform of the \( N(0,1) \) density and it converges uniformly on compact \( C \subset \mathbb{R} \setminus \{0\} \) to \(-1\)

\[ x \to \rho_\ell(z) \text{ for } z \to 0 \text{ (dotted line), } x > 1 \text{ (solid line)} \]

In the case \( \lambda < 1 \) the regularisation captures already with a. f. p. all point of \( M_N \) and the statements follow easily. For \( \lambda > 1 \) different behaviour is expected as the lowest potential value is close to \( E_0 \) and does not fall within the regularisation. It holds that

\[ M_N \sim M_N' + S_N + o(1) \]

where \( S_N \) is the contribution of \( \arg \min_{x \in \mathbb{N}} \rho_\ell \left( V(\ell) \right) \left( \phi_j \right)^* \cdot x \). The resulting necessity of \( S_N \) of being of constant order then yields the respective statements of the theorem.

References