The Model

Let \( G = (V, E) \) be a connected, infinite graph with uniformly bounded degree \( \mathcal{N} \in \mathbb{N} \). It is an easy exercise to establish that the latter property is equivalent to the existence of a \( \mu > 0 \) such that
\[
\sup_{x \in G} \sum_{y \in G} e^{-\mu d(x,y)} < \infty,
\]
where \( d(\cdot, \cdot) \) denotes the natural graph distance on \( G \). Further, we pick two families of real-valued random variables \((h^{(p)}_n, \Lambda_n)\) and \((h^{(q)}_n, \Lambda_n)\). Suppose there is an exhaustive sequence of finite sets \( \Lambda_n \uparrow V \) such that the matrices \( h^{(p)}_n = (h^{(p)}_{n,p})_{p \in \Lambda_n} \); \( t \in [p,q] \) are almost surely symmetric and positive definite. Moreover, we assume the uniform norm bound
\[
\sup_{n \in \mathbb{N}} \|h^{(p)}_n\|, \|h^{(q)}_n\|, \|h^{(p)}_n\|_{\infty} \leq C \quad \text{a.s.}
\]
for some deterministic \( C < \infty \). On a suitably chosen dense domain within the Hilbert space \( H_n = \bigotimes_{p \in \Lambda_n} L^2(\mathbb{R}, dp) \) we install the usual self-adjoint position and momentum operators \( q = (q_p)_{p \in \Lambda_n} \) and \( p = (p_p)_{p \in \Lambda_n} \), respectively. These operators constitute the finite volume Hamiltonian
\[
H_n = (q^TP)^T \frac{\partial}{\partial p} h^{(p)}_n (q^TP).
\]

Entanglement Entropy of Gaussian States

Let \( \Lambda_n \subset V \) be a finite distinguished region and assume \( n \in \mathbb{N} \) large enough such that \( \Lambda_n \subset \Lambda_n \). Then for a density operator \( \sigma_n \in B(H_n) \), the entanglement entropy with respect to the bipartition
\[
H_n = \bigotimes_{p \in \Lambda_n} L^2(\mathbb{R}, dp) \oplus \bigotimes_{p \in \Lambda_n} L^2(\mathbb{R}, dp)
\]
(3)

is given by
\[
S(\sigma_n; \Lambda_n) = - \text{tr} (\sigma_n \log \sigma_n),
\]
where \( \sigma_n = \text{tr}_{n \setminus \Lambda_n} (\sigma_n) \).

It is a well-known fact that the unique ground state of the finite volume Hamiltonian (3) is Gaussian and can be represented by a density matrix, which we shall denote by \( \rho_n \).

Introducing the shorthand \( r = q^T p \), its entanglement entropy with respect to the bipartition (3) can be computed in terms of the covariance matrix \( \Gamma_n \)
\[
\Gamma_n = \text{tr} (\rho_n (x^2, y^2)), \quad x, y = 1, \ldots, |\Lambda_n|,
\]
with the help of the handy formula
\[
S(\rho_n; \Lambda_n) = \sum_{(\gamma_1, \gamma_2) \geq 0} \left( \frac{2}{\gamma_1} - 1 \right) \log \left( \frac{2}{\gamma_1} \right) - \frac{2}{\gamma_1 \gamma_2} \log \left( \frac{2}{\gamma_1} \right),
\]
(4)

where \( \Gamma \) is obtained from \( \Gamma \) by erasing the rows and columns belonging to \( \Lambda_n \setminus \Lambda_n \).

Here, \( \text{sym} \) denotes the symplectic counting multiplicities, which is in turn characterized by the following theorem due to Williamson [3].

**Theorem** Let \( \Gamma \in \mathbb{R}^{2n \times 2n} \) be symmetric and positive definite and define a symplectic form on \( \mathbb{R}^{2n \times 2n} \) by
\[
\Omega = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\]

Then there exists a symplectic matrix \( S \in \text{SP}(2n, \mathbb{R}) = \{ S \in \mathbb{R}^{2n \times 2n} \mid S^T \Omega S = \Omega \} \) such that
\[
S^T \Omega S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.
\]

The Idea of the Proof of the Violation of an Area Law

In either case \( V \in (\mathbb{N}, Z) \), we exploit the fact that the spectral decomposition of the triagonal matrix \( h^{(p)}_n \) is known explicitly and that the resulting sums can be well approximated by Riemann integrals for large \( |\Lambda_n| \).

\( V \in (\mathbb{N}, Z) \). Here, we solve the aforementioned integrals and establish a divergent lower bound on the last diagonal entry of the matrix product whose ordinary spectrum constitutes the entanglement entropy in light of (4) and Williamson’s theorem. The min-max theorem then implies the divergence of the largest eigenvalue, and hence, the divergence of the largest symplectic eigenvalue.

\( V \in (\mathbb{N}, Z) \). Due to the emergence of translation invariance of the system in the limit \( \Lambda_n \uparrow \mathcal{N} \), the limiting operators are Toeplitz. Hence, after deriving a lower bound on the entanglement entropy (4) in terms of determinants of Toeplitz matrices, we have a tremendous arsenal of powerful machinery at our disposal. In particular, we appeal to Szegö’s strong limit theorem to establish the divergence of the determinants in the lower bound.

Eventually, a thorough error analysis furnishes explicit exhaustive sequences of finite sets for which we may infer the asserted divergence of the entanglement entropy.

**References**

