Johnson-Lindenstrauss embeddings are random maps from $\mathbb{R}^n$ to an $\mathbb{R}^m$ of lower dimension that preserve the norms of $p$ (finitely many) given points. This approach can improve the computational complexity of problems by reducing the dimension of the containing space. Thus, we aim to minimize the dimension of the reduced space as well as the computation time for the transformation itself. Currently existing fast constructions either have an upper bound for the number of points $p$ to preserve or an embedding dimension that is not optimal. We present a new construction for a Johnson-Lindenstrauss embedding that has an optimal embedding dimension and also a significantly higher upper bound for $p$. However, it accomplishes a fast transformation for all $p$ points together rather than for each single point.

**Goals for the Construction of JLEs**

As many applications benefit from a lower computational time due to the reduced data dimension, we pursue reaching a low embedding dimension along with a fast computation of the dimension reduction. Our goals are thus:

1. **Optimal embedding dimension**: This has been proven to be $\Theta(n^{1-\epsilon})$ [6], this is independent of $N$.
2. **Fast transformation of a single point**: We aim to transform each single point with the Johnson-Lindenstrauss embedding in time $O(N \log N)$ with fixed $k$.
3. **Fast transformation of all $p$ points in $E$ together**: We aim to transform all points from the set $E$ in an overall time of $O(pN\log N^p)$.

The third goal is a weaker version of the second one. However, it will turn out to be easier to achieve in some cases.

**Existing Constructions for JLEs**

All of the following existing Johnson-Lindenstrauss embeddings can achieve a transformation of a single point in the desired time of $O(N \log N)$. However, there is an upper bound for the number $p$ of points to be preserved and not all of them have the optimal embedding dimension.

**Construction** | **Optimal Dimension** | **Upper Bound for $p$**
--- | --- | ---
Bernoulli + [1] | $\Theta(\frac{1}{\epsilon^2} \log p)$ | $\Theta(p^{1/\epsilon} \log N)$
Ailon, Liberty (2009) [5] | yes | $\exp(\Theta(N^{\epsilon^2}))$
Krahmer, Ward (2011) [2] | no | (only a general bound)

The first construction simply uses independent entries, each taking the value $\pm 1$ with probability $\frac{1}{2}$. The other two approaches are based on the discrete Fourier transform. The given bound for Ailon, Liberty 2009 works for any fixed $\delta > 0$. Based on the minimal embedding dimension, one can show that all Johnson-Lindenstrauss embeddings have the upper bound of $p \leq \exp(O(N))$ for the number of points.

This means, for JLEs of optimal embedding dimension, there is still a gap for the values $p = \exp(O(N^{\epsilon}))$ up to $p = \exp(O(N^{1/\epsilon}))$ between the construction of Ailon, Liberty and the general upper bound. We intend to close this gap with a new construction.

**A New Construction for a Fast JLE I**

Our new construction is based on the following lemma, stating that the composition of two JLEs is a JLE again with slightly larger parameter values.

**Lemma** (Composition of JLEs, S. B., Krahmer, 2017). Let $A \in \mathbb{R}^{m_1 \times m}$, $B \in \mathbb{R}^{m_2 \times n}$ be independent random matrices that are both $(p, \epsilon, \eta)$-JLE. Then $AB \in \mathbb{R}^{m_2 \times n}$ is a $(p, \epsilon, \eta)$-JLE.

**A New Construction for a Fast JLE II**

The previous lemma can be used to show the Johnson-Lindenstrauss property for the following matrix

$$GA \in \mathbb{R}^{m \times N}$$

where

- $A \in \mathbb{R}^{m \times N}$ is the construction by Krahmer and Ward [2]. It can map points from the original $\mathbb{R}^n$ into a space of a reduced, but not yet optimal dimension $m$.
- $G \in \mathbb{R}^{m \times m}$ is the construction using Bernoulli $\pm 1$ matrices [1]. This consecutively maps points from the space $\mathbb{R}^n$ into a space of optimal dimension $m$.

For the choices $m = \Theta(\epsilon^{-2} \log 2)$ and $n = \Theta(\epsilon^{-4} \log(N))$ one can prove $G$ and $A$ to be JLEs, so also the composition $GA$ is a JLE. This means, we get the optimal embedding dimension $m$ for fixed $\epsilon$ and thus this achieves our first goal.

As pointed out before, the computation of $A$ is always fast. The matrix $G$ can cause difficulties for the fast transformation. However, due to the preceding dimension reduction by $A$, the matrix $G$ is here smaller than a single complete Bernoulli matrix. Counting all the operations necessary for the transformation of one point, we get

$$O(N \log N) \leq \exp(O(N^{\epsilon^2}))$$

for every fixed $\epsilon > 0$. This result can essentially exhaust the entire range of possible values for $p$.

**Discussion**

As demonstrated, the new construction can slightly improve the upper bound on $p$ required for a fast transformation of a single point. It improves the simultaneous transformation of all $p$ points to a larger extent. However, this new approach also has the following restrictions.

- The constants hidden in the $O$-notation are very large. This can cause problems in practical computations.
- Depending on the application, the simultaneous transformation of all $p$ points might not be useful.

Thus, it is still an important open research problem to construct a JLE that has an optimal embedding dimension and allows a fast transformation of a single point for the remaining values $p = \exp(O(N^{\epsilon}))$ for $\frac{1}{2} \leq r \leq 1$.

**References**


