Lecture 3: Melnikov method through the Hamilton-Jacobi equation for the perturbed pendulum

Multiscale Phenomena in Geometry and Dynamics
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22-29 July 2019
The problem

We want to find a way to study the splitting of separatrices of the Hamiltonian:

$$H\left( y, x, \frac{t}{\varepsilon} \right) = \frac{y^2}{2} + (\cos x - 1) + \varepsilon(\cos x - 1) \sin \frac{t}{\varepsilon}$$

As we cannot apply the Melnikov formula directly, we will consider the “regular problem”

- Hamiltonian

  $$H(y, x, \omega t) = \frac{y^2}{2} + (\cos x - 1) + \varepsilon(\cos x - 1) \sin \omega t$$

- Equations

  $$\dot{x} = y$$
  $$\dot{y} = \sin x + \varepsilon \sin x \sin \omega t$$

And we will study the splitting of separatrices of the pendulum “directly” to understand how we obtain the bound of the error.

Later we will go to the exponentially small case that will be $$\omega = \frac{1}{\varepsilon}$$ and see if we can improve the bound of the error.
The perturbed pendulum

- From now on:
  \[ H(y, x, \omega t) = \frac{y^2}{2} + (\cos x - 1) + \epsilon(\cos x - 1) \sin \omega t \]

- Equations
  \[
  \begin{align*}
  \dot{x} &= y \\
  \dot{y} &= \sin x + \epsilon \sin x \sin \omega t
  \end{align*}
  \]

- \((x, y) = (0, 0)\) is a hyperbolic periodic orbit for this system.

- We want to compute the splitting between its invariant manifolds for \(\epsilon \neq 0\) small.
Plan

- Look for suitable analytic parameterizations of the invariant manifolds.

- Analyze the difference of the parameterizations.

- Deduce from this analysis the first order of the difference.

- As in our example the unperturbed separatrix is a graph we look for the perturbed invariant manifolds as graphs.
Parameterizations of the invariant manifolds

- We use that the manifolds are Lagrangian graphs and then they are given by the gradient of a function.
  \[ y = \partial_x S^{u,s}(x, \omega t) \]

- The generating functions \( S^{u,s}(x, \tau) \) satisfy the Hamilton-Jacobi equation (PDE).
  \[ \omega \partial_{\tau} S(x, \tau) + H(x, \partial_x S(x, \tau), \tau) = 0 \]

- Another possibility is to look for \( y = f(x, \tau) \)
  and using that is invariant by the flow we obtain the PDE:
  \[ \omega \partial_{\tau} f(x, \tau) + \partial_x H(x, f(x, \tau), \tau) + \partial_y H(x, f(x, \tau), \tau) \partial_x f(x, \tau) = 0 \]
  then \( f = \partial_x S \).
Parameterizations of the invariant manifolds

- Pendulum example, remember:

\[ H(y, x, \omega t) = \frac{y^2}{2} + \cos x - 1 + \varepsilon (\cos x - 1) \sin \omega t \]

- Equation for \( S^{s,u} \):

\[ \omega \partial_{\tau} S + \left( \partial_x S \right)^2 + \frac{\cos x - 1 + \varepsilon (\cos x - 1) \sin \tau}{2} = 0 \]

- Asymptotic conditions:

\[ \begin{cases} 
\lim_{x \to 0} \partial_x S^u(x, \tau) = 0 \\
\lim_{x \to 2\pi} \partial_x S^s(x, \tau) = 0 
\end{cases} \]
Unperturbed system; Parameterizations of the invariant manifolds

- Unperturbed pendulum:

\[ H_0(y, x) = \frac{y^2}{2} + \cos x - 1 \]

- In this case we know that the manifolds coincide:

\[ y = \partial_x S_0(x, \omega t) \]
Unperturbed system; Parameterizations of the invariant manifolds

- Unperturbed pendulum: \( H_0(y, x) = \frac{y^2}{2} + \cos x - 1 \)
- The Hamilton-Jacobi equation becomes:
  \[
  \omega \partial_\tau S(x, \tau) + H_0(x, \partial_x S(x, \tau)) = 0
  \]
- Equation for \( S_0 \):
  \[
  \omega \partial_\tau S + \frac{(\partial_x S)^2}{2} + \cos x - 1 = 0
  \]
- It has a solution \( S_0 = S_0(x) \) where:
  \[
  \frac{(\partial_x S_0)^2}{2} + \cos x - 1 = 0
  \]
  which gives: \( S_0(x) = \pm \sin(x/2) \).
- It satisfies both asymptotic conditions (is an homoclinic):
  \[
  \lim_{x \to 0} \partial_x S_0(x) = \lim_{x \to 2\pi} \partial_x S_0(x) = 0
  \]
Reparameterization using the time over the separatrix

**exercice:** Parameterization of the pendulum separatrix:

\[ x_0(v) = 4 \arctan(e^v), \quad y_0(v) = \frac{2}{\cosh v}, \quad \begin{pmatrix} \dot{x} = y \\ \dot{y} = \sin x \end{pmatrix} \]

Reparametrize \( x = x_0(v), \quad T^{u,s}(v, \tau) = S^{u,s}(x_0(v), \tau) \)

Equation for \( S(x, \tau) \):

\[ \omega \partial_{\tau} S + \frac{(\partial_x S)^2}{2} + \cos x - 1 + \varepsilon(\cos x - 1) \sin \tau = 0 \]

**exercice:**

- Equation for \( T(v, \tau) \):

\[ \omega \partial_{\tau} T + \frac{\cosh^2 v}{8} (\partial_v T)^2 - \frac{2}{\cosh^2 v} - \varepsilon \frac{2}{\cosh^2 v} \sin \tau = 0 \]

- Asymptotic conditions

\[ \begin{cases} \lim_{\text{Re } v \to -\infty} \cosh v \partial_v T^u(v, \tau) = 0 \\ \lim_{\text{Re } v \to +\infty} \cosh v \partial_v T^s(v, \tau) = 0. \end{cases} \]
If we have a solution of:

$$\omega \partial_\tau T + \frac{\cosh^2 v}{8} (\partial_v T)^2 \ - \ \frac{2}{\cosh^2 v} \ - \ \varepsilon \frac{2}{\cosh^2 v} \ \sin \tau = 0$$

- Will give us parameterizations of the form

$$x = 4 \arctan(e^v), \quad y = \frac{\cosh v}{2} \partial_v T_0^{u,s}(v, \tau).$$

- For the unperturbed pendulum: $\partial_v T_0(v) = \frac{4}{\cosh^2 v}$.

$$(x = 4 \arctan(e^v), \quad y = \frac{2}{\cosh v})$$
splitting of separatrices

We want to study the two solutions $T^u, T^s$ of this equation which give the invariant manifolds and see that they are different

$$\omega \partial_\tau T + \frac{\cosh^2 v}{8} (\partial_v T)^2 - \frac{2}{\cosh^2 v} - \varepsilon \frac{2}{\cosh^2 v} \sin \tau = 0$$
New parameterizations

\[ \omega \partial_\tau T + \frac{\cosh^2 v}{8} (\partial_v T)^2 - \frac{2}{\cosh^2 v} - \varepsilon \frac{2}{\cosh^2 v} \sin \tau = 0 \]

- Unperturbed pendulum: \( \partial_v T_0(v) = \frac{4}{\cosh^2 v} \).
- Perturbed pendulum: \( T^{u,s} = T_0 + T^{u,s} \).
- We expect \( T^{u,s} = \mathcal{O}(\varepsilon) \).

Parameterizations of the invariant manifolds:

\[
\begin{align*}
    x &= 4 \arctan(e^v) \\
y &= \frac{2}{\cosh v} + \frac{\cosh v}{2} \partial_v T^{u,s}(v, \tau)
\end{align*}
\]

- To study the difference between manifolds we have to analyze \( \partial_v T^u - \partial_v T^s \).
We will prove the following **Theorem (Melnikov)**

Fix $V > 0$, then there exists $\varepsilon_0$ such that for $0 < \varepsilon < \varepsilon_0$ we have:

- $T^u(v, \tau)$ is defined for $-\infty < v \leq V$, $\tau \in [0, 2\pi]$.
- $T^s(v, \tau)$ is defined for $-V \leq v < +\infty$, $\tau \in [0, 2\pi]$.
- $\partial_v T^u(v, \tau) - \partial_v T^u(v, \tau) = \varepsilon \partial_v L(v, \tau) + O(\varepsilon^2)$ for $-V \leq v < +V$, $\tau \in [0, 2\pi]$.
- $L(v, \tau)$ is the Melnikov potential:

\[
L(v, \tau) = -\int_{-\infty}^{+\infty} (H_1(x_h(v + s), \tau + \omega s) - H_1(0, \tau + \omega s))ds
= -\int_{-\infty}^{+\infty} \frac{2}{\cosh^2(v + s)} \sin(\tau + \omega s)ds = -\frac{2\pi \omega}{\sinh \frac{\pi \omega}{2}} \sin(\tau - \omega v).
\]

- Remember the parameterizations of the invariant manifolds:

\[
\begin{align*}
x &= 4 \arctan(e^v) \\
y &= \frac{2}{\cosh v} + \frac{\cosh v}{2} \partial_v T^u_s(v, \tau)
\end{align*}
\]

- Therefore the distance between the stable and unstable manifolds is given by:

\[
d(v, \tau) = \varepsilon \frac{\cosh v}{2} \partial_v L(v, \tau) + O(\varepsilon^2) = \varepsilon \frac{\pi \omega^2}{\sinh \frac{\pi \omega}{2}} \cosh v \cos(\tau - \omega v) + O(\varepsilon^2)
\]

for $-V \leq v < +V$, $\tau \in [0, 2\pi]$.  

\[ \omega \partial_\tau T + \frac{\cosh^2 \nu}{8} (\partial_\nu T)^2 - \frac{2}{\cosh^2 \nu} - \varepsilon \frac{2}{\cosh^2 \nu} \sin \tau = 0 \]

- Plug \( T^{u,s} = T_0 + T^{u,s} \) into the Hamilton-Jacobi equation.

- Recall \( \partial_\nu T_0(\nu) = \frac{4}{\cosh^2 \nu} \).

- New equation (exercise):

\[ \omega \partial_\tau T + \partial_\nu T + \frac{\cosh^2 \nu}{8} (\partial_\nu T)^2 - \varepsilon \frac{2}{\cosh^2 \nu} \sin \tau = 0 \]

- \( T^{u,s} \) are expected to be small (order \( \varepsilon \)).

- We can solve this equation by inverting the linear differential operator \( \mathcal{L}_0 = \omega \partial_\tau + \partial_\nu \).
The integral operators

- Inverses of $\mathcal{L}_0 = \omega \partial_\tau + \partial_v$. ($\mathcal{L}_0 g = h$, $g = \mathcal{G}(h)$.)

- For functions which decay to zero as $\Re v \to -\infty$,

  $$\mathcal{G}^u(h)(v, \tau) = \int_{-\infty}^{0} h(v + t, \tau + \omega t) \, dt.$$  

- For functions which decay to zero as $\Re v \to +\infty$,

  $$\mathcal{G}^s(h)(v, \tau) = \int_{+\infty}^{0} h(v + t, \tau + \omega t) \, dt.$$
The fixed point equation

- We look for $T^u$ as a solution of
  \[ T^u = G^u \left( -\frac{\cosh^2 v}{8} (\partial_v T^u)^2 + \varepsilon \frac{2}{\cosh^2 v} \sin \tau \right) \]
  with
  \[ G^u(h)(v, \tau) = \int_{-\infty}^{0} h(v + t, \tau + \omega t) \, dt. \]

- We look for $T^s$ as a solution of
  \[ T^s = G^s \left( -\frac{\cosh^2 v}{8} (\partial_v T^s)^2 + \varepsilon \frac{2}{\cosh^2 v} \sin \tau \right) \]
  with
  \[ G^s(h)(v, \tau) = \int_{+\infty}^{0} h(v + t, \tau + \omega t) \, dt. \]
The fixed point equation

- $T^u$ is a fixed point of

$$T^u = F^u(\partial_v T) = G^u \left( -\frac{\cosh^2 v}{8} (\partial_v T)^2 + \varepsilon \frac{2}{\cosh^2 v} \sin \tau \right)$$

- $T^s$ is a fixed point of

$$T^s = F^s(\partial_v T) = G^s \left( -\frac{\cosh^2 v}{8} (\partial_v T)^2 + \varepsilon \frac{2}{\cosh^2 v} \sin \tau \right)$$

Therefore we need to solve these two fixed point equations and then compute $T^u(v, \tau) - T^s(v, \tau)$ (and $\partial_v T^u(v, \tau) - \partial_v T^s(v, \tau)$):
The fixed point argument

\[ T = F^u(\partial_v T) = G^u \left( -\frac{\cosh^2 v}{8} (\partial_v T)^2 + \varepsilon \frac{2}{\cosh^2 v} \sin \tau \right) \]

We solve the equation for the unstable manifold (The stable one is analogous)

- As we want to deal with analytic functions we will take:
  \( \text{Re } v \leq V, \ |\text{Im } v| \leq \rho, \ \text{Re } \tau \in [0, 2\pi], \ |\text{Im } \tau| \leq \sigma \), where \( \rho \) and \( \sigma \) are small.

- We will consider \( \chi \) the (Banach) space of functions \( T(v, \tau) \), \( 2\pi \)-periodic in \( \tau \) and analytic in the previous domain.

- Then \( F^u \) is a function that acts in this (Banach) space.

- We will need a good norm in \( \chi \) which controls the exponential decay at \( -\infty \) of the function \( T^u \).

- As we expect the fix point to be small a good approximation to begin is use 0.
The fixed point argument

\[ T = \mathcal{F}^u(\partial_v T) = \mathcal{G}^u \left( -\frac{\cosh^2 v}{8} (\partial_v T)^2 + \varepsilon \frac{2}{\cosh^2 v} \sin \tau \right) \]

- The equation has a problem, the operator \( \mathcal{F}^u \) acts in the derivatives of \( T \)
- We differentiate both sides with respect to \( v \):

\[ \partial_v T = \partial_v \mathcal{F}^u(\partial_v T) = \partial_v \mathcal{G}^u \left( -\frac{\cosh^2 v}{8} (\partial_v T)^2 + \varepsilon \frac{2}{\cosh^2 v} \sin \tau \right) \]

- We will solve an equation whose unknown will be \( h^u = \partial_v T^u \):

\[ h = \partial_v \mathcal{F}^u(h) = \partial_v \mathcal{G}^u \left( -\frac{\cosh^2 v}{8} h^2 + \varepsilon \frac{2}{\cosh^2 v} \sin \tau \right) \]

- We can recover \( T^u(v, \tau) = \mathcal{F}^u(h^u) \).
The fixed point argument

We look for $h^u$ such that is a fixed point of $h = \partial_v F^u(h)$.

$$\partial_v F^u(h)(v, \tau) = \partial_v G^u \left( -\frac{\cosh^2 v}{8} (h(v, \tau))^2 + \varepsilon \frac{2}{\cosh^2 v} \sin \tau \right)$$

$$\partial_v F^u(0)(v, \tau) = \partial_v G^u \left( \varepsilon \frac{2}{\cosh^2 v} \sin \tau \right) = \varepsilon \partial_v \int_{-\infty}^{0} \frac{2}{\cosh^2(v + t)} \sin (\tau + \omega t) \, dt$$

$$\partial_v F^u(0)(v, \tau) = \varepsilon \int_{-\infty}^{0} \frac{-4 \sinh(v + t)}{\cosh^3(v + t)} \sin (\tau + \omega t) \, dt$$

As in our domain (exercise): $\left| \frac{4 \sinh v}{\cosh^3 v} \right| \leq K e^{2\Re v}$, we have that:

$$|\partial_v F^u(0)(v, \tau)| \leq \bar{K} \varepsilon e^{2\Re v}$$

The constant $\bar{K}$ does not depend on $\varepsilon$. 
The fixed point argument: weighted norms

- Now we choose the norm in $\chi$ to control the function and its decreasing at $-\infty$:
  \[ ||h|| = \sup |h(\nu, \tau)e^{-2\Re \nu}| \]
  where the supremum is taken in our domain.

- Then is equivalent that:
  \[ ||h|| \leq M \text{ to } |h(\nu, \tau)| \leq Me^{2\Re \nu}. \text{ (as } \Re \nu \rightarrow -\infty) \]

- **Exercice** There exist constants $K_1$, $K_2$ such that for $|\Im \nu| \leq \rho$ and $\Re \nu \leq V$, we have
  \[ k_1 e^{-\Re \nu} \leq |\cosh \nu| \leq K_2 e^{-\Re \nu} \quad \left( \frac{1}{K_2} e^{\Re \nu} \leq \frac{1}{|\cosh \nu|} \leq \frac{1}{K_1} e^{\Re \nu} \right) \]

- Therefore is equivalent to use the norm:
  \[ ||h|| = \sup |h(\nu, \tau)\cosh^2 \nu| \]
The fixed point argument: weighted norms

We look for \( h^u \) such that is a fixed point of \( h = \partial_v \mathcal{F}^u(h) \).

\[
\partial_v \mathcal{F}^u(h)(v, \tau) = \partial_v \mathcal{G}^u \left( -\frac{\cosh^2 v}{8} (h(v, \tau))^2 + \varepsilon \frac{2}{\cosh^2 v} \sin \tau \right)
\]

- We had that: \( |\partial_v \mathcal{F}^u(0)(v, \tau)| \leq \tilde{K} \varepsilon e^{2\text{Re } v} \)
- Equivalently: \( |\partial_v \mathcal{F}^u(0)(v, \tau)| \leq \varepsilon \frac{\tilde{K}}{\cosh^2 v} \)
- Clearly we have seen that \( ||\partial_v \mathcal{F}^u(0)|| \leq \tilde{K} \varepsilon \)
- Take \( B_R(0) \) the ball of radius \( R \) in the space \( \chi \), where \( R = 2 \tilde{K} \varepsilon \).
- We have seen that \( \partial_v \mathcal{F}^u(0) \in B_R(0) \)
- Take \( h_1, h_2 \in B_R(0) \), then, as \( \mathcal{G}^u \) is linear:
  \[
  \partial_v \mathcal{F}^u(h_1) - \partial_v \mathcal{F}^u(h_2) = \partial_v \mathcal{G}^u \left( -\frac{\cosh^2 v}{8} (h_1^2 - h_2^2) \right)
  \]

We need to study a little the operator \( \partial_v \mathcal{G}^u \)
the operator $\partial_v G^u$

**Lemma**
Take $g(v, \tau) \in \chi$, then, $\partial_v G^u(g) \in \chi$ and there exists a constant $M > 0$ such that:

$$||\partial_v G^u(g)|| \leq M||g||$$

- To see this we need to bound the derivative: $\partial_v G^u(g) = \partial_v \int_{-\infty}^{0} g(v + t, \tau + \omega t)dt$:

- **Exercice**: There exists a constant $M > 0$ such that, for $|\text{Im } v| \leq \rho$ and $\text{Re } v \leq V$

$$|\cosh^2 v| \int_{-\infty}^{0} \frac{1}{|\cosh^2 (v + t)|} dt \leq M$$

- It is easy to bound $G^u(g)$

$$|G^u(g) \cosh^2 v| \leq |\cosh^2 v| \int_{-\infty}^{0} \frac{1}{\cosh^2 (v + t)} \cosh^2 (v + t) g(v + t, \tau + \omega t)dt|$$

$$\leq \sup |\cosh^2 v g(v, \tau)||\cosh^2 v| \int_{-\infty}^{0} \frac{1}{|\cosh^2 (v + t)|} dt \leq M||g||$$

- This bound is not good because it does not give bounds of $\partial_v G^u(g)$ in terms of $g$

- We need to use that $g$ is periodic to obtain the bounds and change a little the domain to work in sectors instead of strips.
Exercice: If we write $g$ in Fourier series: $g(v, \tau) = \sum g[k](v)e^{ik\tau}$, one has that:

$G(v, \tau) := G^u(g)(v, \tau)$ is given by

$$G(v, \tau) = G^u(g)(v, \tau) = \sum_{k \in \mathbb{Z}} G[k](v)e^{ik\tau}$$

where

$$G[k](v) = \int_{-\infty}^{0} g[k](v + t)e^{ik\omega t} dt$$

which is integrable for $g \in \chi$.

In order to prove the lemma we want bounds of $\frac{dG[k]}{dv}$ in terms of $g[k]$. First we give “good” bounds of $G[k]$. These bounds are computed in different ways depending on the domain and whether $k \neq 0$ or $k = 0$.

We change the domain of $v$. Take $V > 0$ and $0 < \beta_0 < \frac{\pi}{4}$. New domain: $\text{Re } v \leq V$ and $|\text{Im } v| \leq \tan \beta_0 |\text{Re } v|$.

We will use the Fourier norm: $\|g(v, \tau)\| = \sum_{k \in \mathbb{Z}} \|g[k]\| e^{k|\sigma|}$. 
The operator $\partial_v G^u$

First we bound $G^{[k]}$:

$$\cosh^2 v \cdot G^{[k]}(v) = \cosh^2 v \int_{-\infty}^{0} g^{[k]}(v + t) \cdot e^{ik\omega t} dt.$$  

Since the integrand has exponential decay for $\text{Re} \ t \to -\infty$, we change the integration path:

- to the line from 0 to $-\infty$ with angle $+\beta_0/2$, when $k > 0$
- to the line from 0 to $-\infty$ with angle $-\beta_0/2$, when $k < 0$

Writting $t = \xi e^{\mp i\beta_0/2}$:

$$| \cosh^2 u \cdot G^{[k]}(v) | \leq M ||g^{[k]}|| \int_{-\infty}^{0} |e^{ik\omega \xi e^{\mp i\beta_0/2}} \cdot e^{\mp i\beta_0/2}| d\xi$$

$$\leq M ||g^{[k]}|| \int_{-\infty}^{0} |e^{-|k|\omega \xi \sin(\beta_0/2)}| d\xi$$

$$\leq M \frac{||g^{[k]}||}{|k\omega| \sin(\beta_0/2)} \leq \tilde{M} \frac{1}{|k|\omega} ||g^{[k]}||.$$  

and we obtain:  

$$||G^{[k]}|| \leq \tilde{M} \frac{1}{|k|\omega} ||g^{[k]}||, \text{ if } k \neq 0$$
The operator $\partial_v G^u$

Now we can bound $\partial_v G^u(g) = \partial_v G = \sum_{k \in \mathbb{Z}} \frac{d}{dv} G[k](v) e^{ik\tau}$.

\[
\frac{d}{dv} G[k](v) = \int_0^0 \frac{d}{dv} g[k](v + t)e^{ik\omega t} dt
\]
\[
= \int_{-\infty}^0 \frac{d}{dt} (g[k](v + t)e^{ik\omega t}) dt - ik\omega \int_{-\infty}^0 g[k](v + t)e^{ik\omega t} dt
\]
\[
= g[k](v) - \lim_{t \to -\infty} (g[k](v + t)e^{ik\omega t}) - ik\omega G[k](v)
\]
\[
= g[k](u) - ik\omega G[k](v).
\]

Hence, it is clear that for $k \neq 0$, $\left\| \frac{d}{dv} G[k] \right\| \leq (1 + \tilde{M})\|g[k]\|$. 

Moreover, recalling that $\frac{d}{dv} G[0](v) = g[0](v)$, we obtain

\[
\|\partial_v G^u(g)\| = \|\partial_v G\| = \sum_{k \in \mathbb{Z}} \left\| \frac{d}{dv} G[k] \right\| e^{|k|\sigma_i} \leq \tilde{M}\|g\|.
\]
The fixed point argument

Remember the equation: \( h = \partial_v F^u(h) = \partial_v G^u \left( -\frac{\cosh^2 v}{8} h^2 + \varepsilon \frac{2}{\cosh^2 v} \sin \tau \right) \)

1. \( \|\partial_v F^u(0)\| \leq \tilde{K} \varepsilon \)
2. Take \( B_R(0) \) the ball of radius \( R \) in this space, where \( R = 2\tilde{K} \varepsilon \).
3. We have seen that \( \partial_v F^u(0) \in B_R(0) \)
4. Take \( h_1, h_2 \in B_R(0) \), then

\[
\|\partial_v F^u(h_1) - \partial_v F^u(h_2)\| = \|\partial_v G^u \left( -\frac{\cosh^2 v}{8} (h_1^2 - h_2^2) \right)\| \leq \| \frac{\cosh^2 v}{8} (h_1^2 - h_2^2)\| \\
\leq \sup | \cosh^4 v (h_1 + h_2)(h_1 - h_2)| \leq \|h_1 + h_2\| \|h_1 - h_2\| \leq 2R \|h_1 - h_2\| \leq 4\tilde{K} \varepsilon \|h_1 - h_2\|
\]

Therefore, if \( 0 < \varepsilon < \varepsilon_0 = \frac{1}{8\bar{K}} \) the map \( \partial_v F^u \) is a contraction from \( B_R(0) \) in \( B_R(0) \) and it has a unique fixed point \( h^u \in B_R(0) \), that is, \( \| h^u \| \leq 2\tilde{K} \varepsilon \).
Moreover the fix point $h^u$ satisfies:

\[ h^u = \partial_v F^u(h^u) = \partial_v F^u(0) + \partial_v F^u(h^u) - \partial_v F^u(0) \]

\[ \| h^u - \partial_v F^u(0) \| = \| \partial_v F^u(h^u) - \partial_v F^u(0) \| \leq 4\tilde{K}\varepsilon \| h^u \| \leq 8\tilde{K}^2\varepsilon^2 \]

Therefore for $-\infty < v \leq V$:

\[ h^u(v, \tau) = \partial_v F^u(0)(v, \tau) + O(\varepsilon^2) \]
difference between the manifolds

- We have seen that there is \( h^u \) solution of:
  \[
  h^u(v, \tau) = \partial_v F^u(h^u)(v, \tau) = \partial_v G^u \left( -\frac{\cosh^2 v}{8} (h^u(v, \tau))^2 + \varepsilon \frac{2}{\cosh^2 v} \sin \tau \right)
  \]
  such that: \( h^u(v, \tau) = \partial_v F^u(0)(v, \tau) + O(\varepsilon^2) \), for \( v \leq V \)

- Analogously, there is a solution \( h^s \) solution of:
  \[
  h^s(v, \tau) = \partial_v F^s(h^s)(v, \tau) = \partial_v G^s \left( -\frac{\cosh^2 v}{8} (h^s(v, \tau))^2 + \varepsilon \frac{2}{\cosh^2 v} \sin \tau \right)
  \]
  such that: \( h^s(v, \tau) = \partial_v F^s(0)(v, \tau) + O(\varepsilon^2) \), for \( v \geq -V \)

- Now we can compute the difference at \( v \), such that \( v \in \mathbb{R}, -V \leq v \leq V \):
  \[
  h^u(v, \tau) - h^s(v, \tau) = \partial_v F^u(0)(v, \tau) - \partial_v F^s(0)(v, \tau) + O(\varepsilon^2)
  \]
  \[
  = \varepsilon \partial_v \int_{-\infty}^{+\infty} \frac{2}{\cosh^2(v + t)} \sin(\tau + \omega t) \, dt + O(\varepsilon^2)
  \]
  \[
  = \varepsilon \partial_v L(v, \tau) + O(\varepsilon^2) = \varepsilon M(v, \tau) + O(\varepsilon^2)
  \]

- As \( h^u = \partial_v T^u \) and \( h^s = \partial_v T^s \), this concludes the proof of the theorem.
In the case $\omega = \frac{1}{\varepsilon}$ all the proof works the same.

The difference at $\nu$, such that $\nu \in \mathbb{R}, -V \leq \nu \leq V$:

$$\partial_{\nu} T^u(\nu, \tau) - \partial_{\nu} T^s(\nu, \tau) = \varepsilon \partial_{\nu} \int_{-\infty}^{+\infty} \frac{2}{\cosh^2(\nu + t)} \sin \left( \tau + \frac{t}{\varepsilon} \right) dt + O(\varepsilon^2)$$

$$= \varepsilon \partial_{\nu} L(\nu, \tau) + O(\varepsilon^2) = \varepsilon M(\nu, \tau) + O(\varepsilon^2)$$

But now $L(\nu, \tau) = -4\pi \frac{1}{\varepsilon} e^{-\frac{\pi}{2\varepsilon}} (1 + O(e^{-\frac{\pi}{\varepsilon}})) \sin(\tau - \frac{\nu}{\varepsilon})$.

How can we improve the results?