

TOPMATH - ESSAY THEMES

Applicants for summer semester 2020 and winter semester 2020/21 must select one of the following essay themes

1. Quadratic reciprocity law

The quadratic reciprocity law is one of the gems of elementary number theory, the first complete proof is due to Gauss. An element a in \mathbb{Z} is called a quadratic residue modulo some n in \mathbb{N} if it is congruent modulo n to a perfect square. Let p and q be distinct odd primes. The quadratic reciprocity law relates the condition on p to be a quadratic residue modulo q to the condition on q to be a quadratic residue modulo p by a simple and elegant equation.

In your essay, you should state this famous theorem and one of its proofs. In addition you may explain generalizations, applications, and/or related results.

2. Recurrence and Transience of Random Walks on Trees

Given an infinite, connected graph G with vertex set V and edge set E , where each vertex has a finite degree, one can define simple random walk on G as follows. Fix a vertex v in V and let the random walk start at v at time 0. When the random walk is at a vertex w at time n , it chooses its state at time $n + 1$ by choosing a vertex uniformly among the neighbours of w . All these choices are independent. The random walk is *recurrent* if it returns, with probability 1, to v , and it is *transient* if the probability to return to v is strictly less than 1. In fact, a recurrent random walk returns, with probability 1, infinitely often to its starting point while a transient random walk returns, with probability 1, only finitely often. If G is the d - dimensional lattice, a famous theorem by George Polya says that simple random walk is recurrent if and only if $d \in \{1, 2\}$. What about simple random walk on infinite trees? (A tree is a graph without cycles. We assume that the tree is connected and that each vertex has only finitely many neighbours). There is a notion of *branching number* for trees, going back to Russell Lyons. In the special case of a d -regular tree, where each vertex has d neighbours, the branching number is $d - 1$. You should prove in the essay that simple random walk on the tree is transient if the branching number is strictly larger than 1. Note that the tree does not need to be regular. Then you can look at examples, in particular at spherical symmetric trees. You can also compare the branching number with the (lower and upper) exponential growth of the tree or consider biased random walks instead of the simple random walk.

3. Arnold's Cat Map

The goal of the presentation should be twofold. First, you have to introduce Arnold's Cat Map and explain briefly its geometric interpretation and some of its most important properties (you have to decide what these are, i.e., be selective and mind the presentation time limit). In the second part of the presentation, you have to prove *one* out of the following two properties of Arnold's Cat Map: either you prove that the map is ergodic (you may state any of the equivalent characterizations of ergodicity for your proof), or you prove that the map is topologically transitive.

4. The Devil's staircase and the fundamental theorem of calculus

The Cantor ternary function (also known as Devil's staircase) is a famous counterexample to a possible naive extension of the fundamental theorem of calculus (FTC). In your essay you should give a proof of the following FTC:

Let $I \subset \mathbb{R}$ and $u : I \rightarrow \mathbb{R}$. The function u belongs to $AC_{\text{loc}}(I)$ (the space of functions locally absolutely continuous in I) if and only if it holds:

1. u is continuous in I
2. u is differentiable \mathcal{L}^1 -a.e. in I and $u' \in L^1_{\text{loc}}(I)$
(\mathcal{L}^1 stands for the 1-dimensional Lebesgue measure and $L^1_{\text{loc}}(I)$ for the space of functions locally summable in I)
3. the fundamental theorem of calculus is valid: for all $x, y \in I$,

$$u(y) = u(x) + \int_x^y u'(t) dt$$

In addition, you should iteratively construct the Cantor function, discuss its optimal (Hölder) continuity and compare it with the continuity assumptions of the FTC.

If you have any questions please do not hesitate to contact us via e-mail at topmath@ma.tum.de