On the Volume of Convex Bodies
Complexity and Algorithms
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Abstract
The \#P-hard problem of VOLUME COMPUTATION for convex bodies is introduced and the original randomised algorithm by Dyer, Frieze and Kannan solving this problem is presented, including a recent improvement by Cousins and Vempala.

The Problem
Definition 1 (Volume of a convex body)
For a convex (full-dimensional) body \( K \) \( \text{vol}_d(K) \) denotes the Lebesgue measure of \( K \).

Definition 2 (VOLUME COMPUTATION)
Input: A positive integer \( n \), a convex body \( K \).
Task: Compute the volume \( \text{vol}_n(K) \) of \( K \).

In general, VOLUME COMPUTATION is \#P-hard if \( n \) is part of the input.

Even deterministic approximation of the volume is \#P-hard. To deal with this complexity, the problem on the right hand side can be considered instead.

Alternative Problem
Definition 3 (EXPECTED VOLUME COMPUTATION)
Input: \( p \in (0, 1], \varepsilon > 0 \), positive integer \( n \) and a convex body \( K \).
Task: Find an approximation \( \hat{v} \) such that
\[
\text{Prob} \left( \left| \hat{v} - \text{vol}_n(K) \right| > \varepsilon \right) \leq p.
\]
Equivalently, compute \( \hat{v} \) such that with probability at least \( 1 - p \)
\[
1 - p \leq \frac{\text{vol}_n(K)}{\hat{v}} \leq 1 + p.
\]

The Dyer, Frieze and Kannan Algorithm
In 1991 Dyer, Frieze and Kannan presented the first polynomial randomised algorithm to solve EXPECTED VOLUME COMPUTATION for a convex body \( K \) given by a membership oracle and well-rounded, i.e. \( B \subseteq K \subseteq \sqrt{n+1}B \). Here, \( B \) is the \( n \)-dimensional unit ball.
The running time is measured in the number of convex problems solved during its execution which is \( O^*(n^3) \).

The Basic Idea
Compute the volume of \( B := RR, R = \sqrt{n+1} \).
Estimate \( \frac{\text{vol}(B)}{\text{vol}(K)} \) with \( r \) via sampling over \( B \).

Example:
\[
n = 2,\ \ \ \text{vol}(B) = (\sqrt{2}/2)^2 = 1/2,\ \ \ \text{vol}(K) = r \cdot 1.75 = 4/3 \approx 1.33.
\]

The Sampling
Aim: take samples of \( K_{n-1} \) to estimate \( r_i \).
- Use a discretization, a dissection of \( \mathbb{R}^n \) into cubes of size \( \delta \).
- Pick a cube with a Markov Chain on cubes intersecting \( K_{n-1} \).
- Take a random sample from this cube

The running time is strongly influenced by the rapidly mixing property of the above Markov Chain.

Rapidly Mixing Property
The rapidly mixing property measures the distance between an ergodic Markov chain after \( t \) steps and its stationary distribution.

The shell-construction
Define:
\[
\rho_i := \max\{\rho(K) \mid 0 < \rho < 1\}, \quad K_i = \rho_i K \cap B \quad \text{for } i = 0, …, K_1 = B \cap K = K.
\]

Compute an estimate \( r_i \) of \( \frac{\text{vol}(K_i)}{\text{vol}(K_{n-1})} \) via sampling. Then
\[
v = \text{vol}(B) \prod_{i=1}^{K_1} \frac{\text{vol}(K_i)}{\text{vol}(K_{n-1})} = \text{vol}(K).
\]

It can be shown that for a suitable choice of \( \rho \), the ratios \( r_i \) can be bounded form below and above. By this, an exponential blow up is prevented.

Development
During the last 25 years, Dyer, Frieze and Kannans algorithm was continuously improved. The currently fastest algorithm by Cousins and Vempala only solves \( O^*(n^3) \) convex programs. Their construction is analogous to the original one and sketched below.

Bibliography